Lecture Feb 5

called this $\mathcal{V}$ for an abstract but prefer this notation

(1) Let $(M, T^{0,1})$ be a CR mfld, with $\dim M = m$. Recall that CR dim $= \dim_{\mathbb{C}} T^{0,1}_p$

Let's denote CR dim $= n$.

Since $T^{0,1} \cap \underbrace{T^{1,0}}_{\overline{T^{0,1}}} = \{0\}$, we have

$$\underbrace{T^{1,0}_p + T^{0,1}_p}_{\dim_{\mathbb{C}} = 2n} \subseteq \underbrace{\mathbb{C} \otimes T_p M}_{\dim_{\mathbb{C}} = m}$$

The difference $d = m - 2n$ is called CR codim.

We note that since $T^{0,1}$ is $\mathbb{C}$-subbundle, it can be locally trivialized, i.e. near every point $\exists\, \bar{Z}_1, \dots, \bar{Z}_n$ local sections (vector fields) s.t. s.t. any local section $\bar{X}$ of $T^{0,1}$ can be rep.

$$\overline{X} = \sum_{j=1}^{n} a_j \overline{Z}_j \quad , \text{ where } a_j \text{ are}$$

smooth functions near the given point.

Sections $\overline{X}$ of $T^{0,1}$ are called

<u>CR vector fields</u> and we refer

to $\overline{Z}_1, \dots, \overline{Z}_n$ as a frame (or basis)

for the CR vector fields.

Of course, if $\overline{Z}'_1, \dots, \overline{Z}'_n$ is another

basis, then $\exists$ $n \times n$ matrix A of

smooth functions s.t.

$$\overline{Z} = A\, \overline{Z}' ,$$

$\overline{Z} = (\overline{Z}_1, \dots, \overline{Z}_n)^t \qquad \overline{Z}' = (\overline{Z}'_1, \dots, \overline{Z}'_n)^t.$

Rem. $Z_1, \dots, Z_n$ is of course then a

frame for $T^{1,0}$

(2) We now consider the case of a CR mfld $M \subseteq \mathbb{C}^N$. In fact, we shall consider the case where M is <u>generic</u> of codim $= d$.

Recall this means M can be defined by d real equations

$$\rho_1 = \dots = \rho_d = 0,$$

where $\partial \rho_1 \wedge \dots \wedge \partial \rho_d \neq 0$.

Let $p \in M$. We shall introduce convenient local coordinates near p, and frame for CR vector fields.

<u>Prop 1.</u> $\exists$ local coord's $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ ($N = n+d$) near p s.t. $p = (0,0)$, and M is given as graph $\operatorname{Im} w_k = \varphi_k(z, \bar z, \operatorname{Re} w)$, where φ_k are smooth functions w/ $\varphi_k(0) = 0$, $d\varphi_k(0) = 0$.

WLOG

<u>Sketch of pf:</u> Let $\rho_1, \dots, \rho_d$ be local defining functions near $p = 0$

Then, $\rho_k = \operatorname{Im}(l_k(z)) + O(2)$,

where $l_k(z) = \sum_{j=1}^{n} v_k^j z_j$. We note

that $\partial\rho_k(0) = \sum_j v_k^j dz_j$ and hence

the $v_k = (v_k^1, \dots, v_k^n)$ are lin. indep

Thus, we can take $w_k = l_k(z)$ as a partial coordinate system. We can now apply Implicit Function Thm to the system of equations $\rho_1 = \dots = \rho_d = 0$ to solve for $\operatorname{Im} w_k$, $k = 1, \dots, d$.

(More precisely $\longrightarrow$

If we let $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_n)$ be some other coordinates s.t. $(\tilde{z}, w)$ is a local coordinate system, then the conclusion of the IFT is that system $\rho_1 = \dots = \rho_d = 0$ is equivalent to

$$\operatorname{Im} w_k = \varphi_k(\tilde{z}, \bar{\tilde{z}}, \operatorname{Re} w)$$

for some smooth functions φ_k.

Since $p = 0$ is a solution, $\varphi_k(0) = 0$.

The fact that $\rho_k = \operatorname{Im} w_k + O(2)$ implies that $d\varphi_k(0) = 0$.

Let's find a convenient frame $\overline{Z}_1, \dots, \overline{Z}_n$ near $p=0$ in the coord's (z,w) given by Prop 1. (These are often called <u>regular coordinates</u>.)

If we identify tangent spaces w/ subspaces of $\mathbb{C}^n \times \mathbb{C}^d$ (as one usually does in calculus) then one easily checks that $T_0 M = \{\operatorname{Im} w = 0\}$, and $T_0^{0,1} M = \{w=0\}$. Thus, it is reasonable to look for the frame in the form

$$\overline{Z}_m = \frac{\partial}{\partial \bar{z}_m} + \sum_{k=1}^{d} \varphi_m^k \frac{\partial}{\partial \bar{w}_k}, \quad m=1,\dots,n$$

Note now that for a $(0,1)$ vector field $\overline{X} = \sum X^j \frac{\partial}{\partial \bar{\zeta}_j}$ $(\zeta = (z, w))$ to be tangent to M given by $\rho_1 = \dots = \rho_d = 0$, the condition

$$\sum_{j=1}^{n} \frac{\partial \rho_k}{\partial \bar{z}_j} X^j = 0, \quad k = 1, \dots, d.$$

can be expressed $\overline{X} \rho_k = 0$.

Thus, to check that $\overline{Z}_i$ are tangent we apply $\overline{Z}_m$ to $\rho_k = -\operatorname{Im} w_k + \varphi_k$

We get

$$\overline{Z}_m \rho_k = \frac{\partial \varphi_k}{\partial \bar{z}_m} + \varphi^k_{\bar{z}_m} \frac{1}{2i} + \sum_{t=1}^{d} \varphi^t_{\bar{z}_m} \frac{\partial \varphi_k}{\partial s^t} \frac{i}{2}$$

where we have written

$$\varphi_k = \varphi(z, \bar{z}, s)$$

Since $\frac{\partial \varphi_n}{\partial s}(0)=0$, this system can be solved for ψ_m^k and can be expressed in matrix form in terms of the $\frac{\partial \varphi_n}{\partial \bar{z}_m}$, $\frac{\partial \varphi_n}{\partial s_t}$ (c.f. Prop 1.6.1 in [BER]).

Let's focus on the case of a hypersurface ($d=1$). Then $(z,w)\in\mathbb{C}^n\times\mathbb{C}$ and

$$\operatorname{Im} w = \varphi(z,\bar{z},\operatorname{Re} w).$$

$$\bar{Z}_m = \frac{\partial \varphi}{\partial \bar{z}_m} + \psi_m\left(\frac{1}{2i}+\frac{1}{2}\frac{\partial \varphi}{\partial s}\right) \Rightarrow$$

$$\psi_m = -2i\,\frac{\varphi_{\bar{z}_m}}{1+i\varphi_s},$$ where

e.g. $\varphi_{\bar{z}_m}$ means $\frac{\partial \varphi}{\partial \bar{z}_m}$.

We have proved

<u>Prop 2</u>. For hypersurface $M \subseteq \mathbb{C}^n \times \mathbb{C}$ in regular coordinates (z, w), a frame for CR v.f. is given by

$$\overline{Z}_m = \frac{\partial}{\partial \overline{z}_m} - 2i \frac{\varphi_{\overline{z}_m}}{1 + i\varphi_s} \frac{\partial}{\partial \overline{w}} .$$